Relativistic extension of the Schrödinger equation no longer requiring the "Dirac sea"

Brian B. K. Min[a)]
Oxford Business Park, 3160 De La Cruz Boulevard, Santa Clara, California 95054, USA

(Received 28 April 2017; accepted 28 September 2018; published online 23 October 2018)

Abstract: A new, relativistic quantum wave equation is obtained by applying the quantum prescriptions to the kinetic energy and momentum instead of the total energy and momentum. This provides the Schrödinger equation with a relativistic extension that modifies the Klein-Gordon and Dirac equations. The wave functions for the modified Klein-Gordon equation are shown to allow the probabilistic interpretation. For a resting particle, the modified Dirac equation gives a true vacuum state in addition to the wave solutions, no longer requiring the "Dirac Sea." © *2018 Physics Essays Publication*. [http://dx.doi.org/10.4006/0836-1398-31.4.421]

Résumé: Une nouvelle équation d'onde quantique relativiste est obtenue en appliquant les prescriptions quantiques à l'énergie cinétique et à la quantité de mouvement au lieu de l'énergie totale et de la quantité de mouvement. Ceci fournit l'équation de Schrödinger avec une extension relativiste qui modifie les équations de Klein-Gordon et de Dirac. Les fonctions d'onde pour l'équation de Klein-Gordon modifiée sont présentées pour permettre l'interprétation probabiliste. Pour une particule au repos, l'équation modifiée de Dirac donne un véritable état de vide en plus des solutions d'ondes, ne nécessitant plus la "mer de Dirac".

Key words: Quantum; Relativistic; Schrödinger; Klein-Gordon; Dirac.

I. INTRODUCTION

It is well known that neither the Klein-Gordon (KG) equation nor the Dirac equation, the two basic relativistic quantum wave equations, reduces to the Schrödinger equation in the nonrelativistic limit. All one has to see is that the energy level for a hydrogenlike atom calculated by either the KG or Dirac equation, when higher order fine structure terms are ignored, differs from that calculated by the Schrödinger equation by the amount of Mc^2, where M is the reduced mass of the electron-nucleus system and c is the speed of light.[1] In this paper, I show a new relativistic quantum wave equation that emerges by requiring that they reduce to the latter. I begin with the familiar energy momentum relation for a particle[2–7]

$$E^2 = P^2c^2 + M^2c^4. \tag{1}$$

If I denote $\mathscr{E} = Mc^2$, the internal energy (many authors call this the rest energy) I can then write $E = \gamma Mc^2 = \gamma\mathscr{E}$ to be the relativistic total energy, $\mathscr{P} = Mv$ to be the nonrelativistic momentum, and $P = \gamma Mv = \gamma\mathscr{P}$ to be the relativistic momentum where $\gamma = 1\Big/\sqrt{1-\dfrac{v^2}{c^2}}$ and v is (the absolute value of) the velocity of the particle. The energy momentum relation, Eq. (1), may then be written

$$\mathscr{E}^2 = \mathscr{P}^2c^2 + \frac{1}{\gamma^2}M^2c^4. \tag{2}$$

When I work with the boosted energy and momentum, E and P as in Eq. (1), I will say, I work in γ-metric, that is I work within

$$1 \le \gamma < \infty.$$

When I work with energy and momentum $\mathscr{E}$ and $\mathscr{P}$ that are not boosted as in Eq. (2), I will say, in the following I work in $1/\gamma$-metric, that is I work within

$$0 \le 1/\gamma \le 1.$$

An advantage of working in $1/\gamma$-metric is that as the velocity of the particle approaches the speed of light, I avoid infinity ($\gamma \to \infty$ when $v \to c$) but work with zero ($1/\gamma \to 0$ when $v \to c$) instead. Another advantage is conceptual; if the particle velocity is c, I no longer have to say, the mass is zero but instead simply the effect of mass is zero. The γ appears in the equation as parameter; for instance, each of the electron's orbits in an atom has a particular angular velocity and radius hence characteristic γ or $1/\gamma$ value, a crucial information about the status of the particle.

II. SCHRÖDINGER EQUATION

I will first show the formalism to obtain the Schrödinger equation in the nonrelativistic limit both in γ-metric and $1/\gamma$-metric. I will then extend the same method to derive a fully relativistic form of the quantum wave equation. I will discuss how the new equation relates to the KG and Dirac equations.[1,8–13]

[a)]bmin@nubron.com

 © 2018 Physics Essays Publication

In γ-metric, I rearrange Eq. (1) to get

$$(E - Mc^2)(E + Mc^2) = P^2c^2 \tag{3}$$

and note that for $v \ll c$, $E + Mc^2 = \gamma Mc^2 + Mc^2 \approx 2Mc^2$. Hence for the low velocity end, I get

$$T \approx \frac{P^2}{2M}, \tag{4}$$

where $T = E - Mc^2$ is the relativistic kinetic energy of the particle.

By substituting $i\hbar\frac{\partial}{\partial t}$ for T, $i\hbar\nabla$ for P, and operating on a function Ψ, I obtain the Schrödinger equation in the free field

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi. \tag{5}$$

In $1/\gamma$-metric, I use the same method as in the above to rearrange Eq. (2) and note that for $v \ll c$, $\mathscr{E} + \frac{Mc^2}{\gamma} \approx 2Mc^2$. Hence for the low velocity end, I obtain

$$\mathscr{T} \approx \frac{\mathscr{P}^2}{2M}, \tag{6}$$

where $\mathscr{T} = \frac{T}{\gamma} = \mathscr{E} - \frac{Mc^2}{\gamma}$. By substituting $i\hbar\frac{\partial}{\partial t}$ for $\mathscr{T}$, $i\hbar\nabla$ for $\mathscr{P}$ and operating on a function Ψ, I again obtain the Schrödinger equation in the free field, Eq. (5).

III. RELATIVISTIC QUANTUM WAVE EQUATION

The above derivation of the Schrödinger equation presents a way of naturally extending the same to the fully relativistic cases both in γ-metric and $1/\gamma$-metric. I note that it is the kinetic energy and momentum that the quantum prescriptions must be applied to, not the total energy and momentum, since after all it is the kinetic energy that generates the momentum.

A. Relativistic extension of the Schrödinger equation in γ-metric

I further rearrange Eq. (1) to read

$$\begin{aligned}(E - Mc^2)(E - Mc^2 + 2Mc^2) = P^2c^2 \quad \text{or} \\ T^2 + 2Mc^2T = P^2c^2.\end{aligned} \tag{7}$$

By substituting $\pm i\hbar\frac{\partial}{\partial t}$ for T, $\pm i\hbar\nabla$ for P (signs in order), and operating on a function Φ, I obtain

$$i\hbar\frac{\partial}{\partial t}\Phi = \pm\frac{\hbar^2}{2M}\Box\Phi, \tag{8}$$

where $\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$or in the tensor notation with the metric $(+\,-\,-\,-)$

$$\partial_\mu\partial^\mu\Phi = \pm 2i\frac{Mc}{\hbar}\partial_0\Phi; \quad \mu = 0, 1, 2, 3\,. \tag{9}$$

B. Relativistic extension of the Schrödinger equation in $1/\gamma$-metric

I further rearrange Eq. (2) in the same manner as Eq. (7), substitute $\pm i\hbar\frac{\partial}{\partial t}$ for $\mathscr{T} \equiv T/\gamma$ and $\pm i\hbar\nabla$ for $\mathscr{P} \equiv P/\gamma$, and operate on a function Φ to obtain

$$i\hbar\frac{\partial}{\partial t}\Phi = \pm\frac{\hbar^2\gamma}{2M}\Box\Phi \tag{10}$$

or

$$\partial_\mu\partial^\mu\Phi = \pm 2i\frac{Mc}{\hbar\gamma}\partial_0\Phi\,. \tag{11}$$

The γ-metric equation, Eq. (9), may be transformed to the above simply by replacing the mass term M with M/γ.

Note that both Eqs. (8) and (10) reduce to the Schrödinger equation, (5), if $\gamma = 1$ and the d'Alembertian $\Box$ is replaced with $-\nabla^2$. The above is a new relativistic quantum wave equation for massive particles that reduces to the Schrödinger equation in the nonrelativistic limit, $\gamma \approx 1$ and $\Box \approx -\nabla^2$.

I define a unit four vector

$$I^\mu = I_0^\mu + I_1^\mu + I_2^\mu + I_3^\mu, \tag{12}$$

where

$$\begin{aligned}I^\mu &= \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad I_0^\mu = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad I_1^\mu = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \\ I_2^\mu &= \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad I_3^\mu = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.\end{aligned} \tag{13}$$

When applied to the four derivative, it is understood that

$$\begin{aligned}&I^\mu\partial_\mu = \partial_0 - \partial_1 - \partial_2 - \partial_3 \\ &I_0^\mu\partial_\mu = \partial_0; \quad I_1^\mu\partial_\mu = -\partial_1; \quad \text{etc.}\end{aligned} \tag{14}$$

This allows the above equation to be rewritten in a more maneuverable form

$$\partial_\mu\partial^\mu\Phi = \pm 2i\frac{Mc}{\hbar\gamma}I_0^\mu\partial_\mu\Phi. \tag{15}$$

In the following, I shall refer to the above as the Modified Klein-Gordon (MKG) equation or sometimes the Min equation (in the $1/\gamma$-metric) at least until we agree to a better name, such as the "Relativistic Schrödinger Equation." It is interesting to note that the left hand side term of the above, the d'Alembertian, is contained in both the Maxwell's and the Klein-Gordon equations and the right hand side (time derivative term) in both the Schrödinger and the Dirac equations, but none of these equations contain both.

The Min equation may be decoupled into the bispinor equations by deploying the Dirac formalism as following.

C. A modified Dirac equation

I will rewrite the energy-momentum equation, Eq. (1), in a tensor form

$$P^\mu P_\mu - M^2c^2 = 0, \tag{16}$$

where $\mu = 0, 1, 2, 3$, and

$$P^\mu = (P^0, P^1, P^2, P^3) = (P^0, P^i) = (E/c, P^i), \tag{17}$$

where $i = 1, 2, 3$. Following Dirac,[9,10,13] the above may be factored into two 4×4 linear algebraic matrix equations

$$P^\mu P_\mu - M^2c^2 = (\gamma^\kappa P_\kappa + Mc)(\gamma^\kappa P_\kappa - Mc), \tag{18}$$

where the Dirac matrices, γ^μ, are defined

$$\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) = (\gamma^0, \gamma^i) \tag{19}$$

with

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{20}$$

I note $P^0 = E/c = \gamma\mathscr{E}/c = \gamma Mc$ and denote $(P_1, P_2, P_3) \equiv \vec{P}$ and $(\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3) \equiv \vec{\mathscr{P}}$, hence $\vec{P} = \gamma\vec{\mathscr{P}}$.

I (or simply 1) is a 2×2 unit matrix, and σ^i are 2×2 Pauli matrices. From Eqs. (16) and (18), I get a factored equation

$$\gamma^\kappa P_\kappa - Mc = 0 \tag{21}$$

and

$$\gamma^\kappa P_\kappa + Mc = 0. \tag{22}$$

From the first set of the factored equations, Eq. (21), I get

$$\gamma^0\mathscr{E} - \frac{M}{\gamma}c^2 - c\sum_i \gamma^i \mathscr{P}^i = 0 \tag{23}$$

which may be further rearranged

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathscr{E} - \frac{M}{\gamma}c^2 - c\sum_i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\mathscr{P}^i = 0, \tag{24}$$

and finally, to

$$\begin{pmatrix} \mathscr{E} - \frac{M}{\gamma}c^2 & 0 \\ 0 & -\left(\mathscr{E} - \frac{M}{\gamma}c^2\right) - 2\frac{M}{\gamma}c^2 \end{pmatrix} = c\sum_i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\mathscr{P}^i. \tag{25}$$

By substituting $\pm i\hbar\partial_t$ for $\mathscr{T} = \mathscr{E} - Mc^2/\gamma = T/\gamma$, $\pm i\hbar\nabla$ for $\mathscr{P} = P/\gamma$ (signs in order), where $\partial_t \equiv \frac{\partial}{\partial t}$, and operating on spinors Ψ_A and Ψ_B, defined by

$$\Psi \equiv \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}; \qquad \Psi_A \equiv \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad \Psi_B \equiv \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix} \tag{26}$$

I obtain

$$\begin{pmatrix} \pm i\hbar\partial_t & 0 \\ 0 & \mp i\hbar\partial_t - 2\frac{M}{\gamma}c^2 \end{pmatrix}\begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = \pm ic\hbar\sum_{i=1,2,3}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\partial_i\begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}. \tag{27}$$

By using the first of the following relationships: $\gamma^0 - 1 = 2\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ and $\gamma^0 + 1 = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, the above may be written,

$$\pm i\hbar\gamma^\mu\partial_\mu\Psi + \left(\gamma^0 - 1\right)\frac{Mc}{\gamma}\Psi = 0. \tag{28}$$

In the following, I will call the above the Modified Dirac (MDirac) equation. In a decoupled form, it reads from Eq. (27)

$$\begin{aligned} \partial_0\Psi_A &= \sigma^i\partial_i\Psi_B \\ \left(\pm\partial_0 - 2i\frac{Mc}{\hbar\gamma}\right)\Psi_B &= \pm\sigma^i\partial_i\Psi_A. \end{aligned} \tag{29}$$

The first is a massless, electromagnetic interaction between the spinors, Ψ_A and Ψ_B. The second is a massive interaction between the two as long as $1/\gamma > 0$. If $v = c$, then $1/\gamma = 0$, and both are massless interactions. Massless in the latter means not $M = 0$, but $M/\gamma = 0$.

From the second set of factored equations, Eq. (22), I get

$$\pm i\hbar\gamma^\mu\partial_\mu\Psi + \left(\gamma^0 + 1\right)\frac{Mc}{\gamma}\Psi = 0. \tag{30}$$

I then get the exact same set of decoupled equations as Eq. (29) except Ψ_A and Ψ_B are interchanged.

Hence, the modified Dirac equation, Eq. (28), is derived from the Min equation. Conversely, the modified Dirac spinors Ψ_A and Ψ_B can be shown to revert to Min equation when combined. The modified Dirac equation describes the interaction of two spinors both with and without mass.

This compares with the Dirac equation

$$i\hbar\gamma^\mu\partial_\mu\Psi - Mc\Psi = 0 \tag{31}$$

which may be decoupled to

$$\begin{aligned}\left(\partial_0+i\frac{Mc}{\hbar}\right)\Psi_A&=\sigma^i\partial_i\Psi_B\\ \left(\partial_0-i\frac{Mc}{\hbar}\right)\Psi_B&=\sigma^i\partial_i\Psi_A.\end{aligned} \tag{32}$$

Both of the above pair of decoupled Dirac equations describe the interaction of the two spinors through mass, compared to Eq. (29) where in addition to the similar pair, a third solution describes a massless interaction.

D. Simple solutions for the modified Dirac equation

1. A particle at rest

If Ψ is independent of position, I get

$$\frac{\partial\Psi}{\partial x}=\frac{\partial\Psi}{\partial y}=\frac{\partial\Psi}{\partial z}=0, \tag{33}$$

i.e., $P_x=P_y=P_z=0$, or zero momentum and zero velocity with $1/\gamma=1$. The Modified Dirac equation, Eq. (28), then reduces to

$$\pm\frac{i\hbar}{c}\gamma^0\frac{\partial\Psi}{\partial t}+\left(\gamma^0-1\right)Mc\Psi=0 \tag{34}$$

or

$$\pm\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\dfrac{\partial\Psi_A}{\partial t}\\ \dfrac{\partial\Psi_B}{\partial t}\end{pmatrix}-\begin{pmatrix}0&0\\0&-2\end{pmatrix}i\frac{Mc^2}{\hbar}\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix}=0. \tag{35}$$

I then get

$$\begin{aligned}\frac{\partial\Psi_A}{\partial t}&=0\\ \frac{\partial\Psi_B}{\partial t}&=\pm i\frac{2Mc^2}{\hbar}\Psi_B\end{aligned} \tag{36}$$

or

$$\begin{aligned}\Psi_A&=\text{constant}\\ \Psi_B&=e^{\pm i\frac{Mc^2}{\hbar/2}t}\Psi_B(0).\end{aligned} \tag{37}$$

The solutions of Ψ_A and Ψ_B may be interchanged due to the second set of equations, Eq. (30). The above particles-at-rest solutions compare with those of Dirac equation which reads[13]

$$\begin{aligned}\Psi_A&=e^{-i\frac{Mc^2}{\hbar}t}\Psi_A(0)\\ \Psi_B&=e^{i\frac{Mc^2}{\hbar}t}\Psi_B(0).\end{aligned} \tag{38}$$

One of the above Dirac pair is a negative energy solution representing antiparticles. The solutions for the MDirac equation, Eq. (37), contain a constant solution in addition to the particle-antiparticle solutions. The constant solution can be set to zero representing the true vacuum state, eliminating the requirement for the "Dirac Sea." The particle solutions have twice the energy of those for the Dirac equation, Eq. (38).

2. Plane wave solutions

For the Modified Dirac equations, Eq. (28), I now look for the plane wave solution of the type

$$\Psi(x)=ae^{\pm i\kappa\cdot x}u(\kappa). \tag{39}$$

Note that

$$\partial_\mu\Psi(x)=\pm i\kappa_\mu\Psi(x)$$

and

$$\gamma^\mu\kappa_\mu=\gamma^0\kappa^0-\vec{\gamma}\cdot\vec{\kappa}=\begin{pmatrix}\kappa^0&-\vec{\kappa}\cdot\vec{\sigma}\\ \vec{\kappa}\cdot\vec{\sigma}&-\kappa^0\end{pmatrix}. \tag{40}$$

The Modified Dirac equation, Eq. (28), becomes

$$\left[\mp\hbar\gamma^\mu\kappa_\mu+\left(\gamma^0-1\right)\frac{Mc}{\gamma}\right]u=0 \tag{41}$$

or

$$\begin{pmatrix}\mp\hbar\kappa^0&\pm\hbar\vec{\kappa}\cdot\vec{\sigma}\\ \mp\hbar\vec{\kappa}\cdot\vec{\sigma}&\pm\hbar\kappa^0-2\dfrac{Mc}{\gamma}\end{pmatrix}\begin{pmatrix}u_A\\ u_B\end{pmatrix}=0, \tag{42}$$

so I get,

$$\begin{aligned}u_A&=\frac{\vec{\kappa}\cdot\vec{\sigma}}{\kappa^0}u_B\\ u_B&=\frac{\vec{\kappa}\cdot\vec{\sigma}}{\kappa^0\mp\dfrac{Mc}{(\hbar/2)\gamma}}u_A.\end{aligned} \tag{43}$$

In the above, u_A and u_B may be interchanged owing to Eq. (30). By the use of the relationship $P^\mu=\gamma\mathscr{P}^\mu=\gamma\hbar\kappa^\mu=\hbar k^\mu$ and $\mathscr{P}^0=E/c=Mc$, I then get

$$\begin{aligned}u_A&=\frac{\mathscr{P}\cdot\vec{\sigma}}{\mathscr{P}^0}u_B\\ u_B&=\frac{\mathscr{P}\cdot\vec{\sigma}}{\mathscr{P}^0\mp\dfrac{2Mc}{\gamma}}u_A.\end{aligned} \tag{44}$$

I can carry the above further to obtain the canonical expressions for u_A and u_B, which, however, I will not pursue here. It suffices to note that the two bispinors, u_A and u_B, interact in the purely electromagnetic manner on one hand

and through mass on the other. For the limiting case, $v = c$ or $1/\gamma = 0$, both are electromagnetic interactions.

IV. DISCUSSIONS

In this section, I discuss the Lorentz covariance of the new formulation, in particular, the MKG equations, Eqs. (9) and (11), and the MDirac equations, Eqs. (28) and (30). Hereinafter, (M)KG stands for either (Modified) Klein-Gordon or (Modified) Klein-Gordon equation(s) and similarly (M) Dirac for (Modified) Dirac or (Modified) Dirac equation(s). The Lorentz covariance of the KG and Dirac equations may be proved directly by the Lorentz transformation.[13,14] By considering time-independent wave functions,[15] the author already showed the MKG to be spontaneously symmetry-broken form of the KG. Here I show it in a more direct way: for both MKG and MDirac via local U(1) gauge transformation of their counterparts, KG and Dirac, respectively, followed by specializing in the time dimension hence breaking the space-time symmetry. It is clear then this symmetry-breaking originates from the use of kinetic energy instead of the total energy for the first quantization (for all four equations).

The second source of the symmetry-breaking is the use of $1/\gamma$-metric to regularize the equations, as in Eqs. (11), (28), and (30). I will address the latter first in Sec. IV A because I can address it in general terms and put it aside for the discussions that follow. The former requires more specific analyses and will be addressed in Secs. IV B and IV C.

Remarkably, the symmetry-broken MKG equations, Eqs. (9) and (11), allow the probability density interpretation of its wave function as discussed in Ref. 15 and briefly repeated below in Sec. IV D. Finally, in Sec. IV E, the same paper is also referenced to show that at least by a generic calculation, the renormalization in the quantum field theory may be rendered unnecessary by the $1/\gamma$-metric formulation.

A. Symmetry breaking in the $1/\gamma$-metric formulation

I will address this in general terms by considering the familiar KG equation

$$\partial_\mu \partial^\mu \Phi + \left(\frac{Mc}{\hbar}\right)^2 \Phi = 0. \tag{45}$$

This equation comes from the familiar energy-momentum relation, Eq. (1), which we can also write as Eq. (16). The second term in Eq. (16) is invariant since both M and c are invariant. The first term is therefore invariant which means Eq. (45) is covariant. On the other hand, the $1/\gamma$-metric form of Eq. (45)

$$\partial_\mu \partial^\mu \Phi + \left(\frac{Mc}{\hbar\gamma}\right)^2 \Phi = 0 \tag{46}$$

comes from the $1/\gamma$-metric form of the energy-momentum relation, Eq. (2), which may be written as

$$\mathscr{P}^\mu \mathscr{P}_\mu = \left(\frac{M}{\gamma}\right)^2 c^2,$$

where $\mathscr{P}^\mu = (\mathscr{P}^0, \mathscr{P}^1, \mathscr{P}^2, \mathscr{P}^3) = (\mathscr{E}/c, \mathscr{P}^i)$. LHS of the above is variant since in RHS M/γ is variant. Equation (46) is variant because its γ-symmetry is spontaneously broken. Equation (2), however, is mathematically equivalent to (1) and its covariance is recovered by simply replacing the mass term M/γ with M. Another possibility is we treat M/γ to be the "proper mass," just like the "proper time" of the special relativity. Equation (46) then is covariant in itself. This consideration applies to all $1/\gamma$-metric formulation. Advantages of working with $1/\gamma$-metric formulation are stated in Sec. I, but some additional accounts have been added below in Sec. IV E.

Note that although in various forms, the energy momentum relations corresponding to the various forms of quantum wave equations all come from the same energy-momentum relation, Eq. (1). The various forms of quantum wave equations and the corresponding forms of the energy-momentum relation are listed below in Table I.

B. Derivation of MKG by a gauge transformation of KG

Here I show that the MKG may be derived directly from KG by a local U(1) gauge transformation $\Phi \to e^{\pm i\theta}\Phi$; $\theta = \theta(x)$and specializing the result in $\theta = \theta(t)$. For direct comparisons with KG, I work with the γ-metric formulation. We first obtain the second derivative of the above transformation

$$\begin{aligned}\partial^\mu \partial_\mu (e^{\pm i\theta}\Phi) = {} & \pm i(\partial^\mu \partial_\mu \theta) e^{\pm i\theta}\Phi + i(\partial^\mu\theta) i(\partial_\mu\theta) e^{\pm i\theta}\Phi \\ & \pm i(\partial_\mu\theta) e^{\pm i\theta}\partial^\mu\Phi \pm i(\partial^\mu\theta)e^{\pm i\theta}\partial_\mu\Phi \\ & + e^{\pm i\theta}\partial^\mu\partial_\mu\Phi.\end{aligned}$$

Plug the above into the KG equation, Eq. (45), we then get

$$\begin{aligned} & \pm i(\partial^\mu\partial_\mu\theta)\Phi - (\partial^\mu\theta)(\partial_\mu\theta)\Phi \pm 2i(\partial^\mu\theta)\partial_\mu\Phi \\ & + \partial^\mu\partial_\mu\Phi + \left(\frac{Mc}{\hbar}\right)^2 \Phi = 0.\end{aligned}$$

This is the local U(1) gauge transformed KG. Now suppose $\theta = \theta(t) = \frac{Mc^2}{\hbar}t = \frac{Mc}{\hbar}x_0$, then

$$\begin{aligned} & \partial^\mu\partial_\mu\theta = \partial^0\partial_0\theta = 0; \\ & (\partial^\mu\theta)(\partial_\mu\theta) = (\partial^0\theta)(\partial_0\theta) = \left(\frac{Mc}{\hbar}\right)^2; \\ & (\partial^\mu\theta)\partial_\mu\Phi = (\partial^0\theta)\partial_0\Phi = \frac{Mc}{\hbar}\partial_0\Phi = \frac{Mc}{\hbar} I_0^\mu \partial_\mu\Phi.\end{aligned}$$

Hence by cancelling out terms, we arrive at

$$\partial_\mu\partial^\mu\Phi \pm i\frac{2Mc}{\hbar} I_0^\mu \partial_\mu\Phi = 0,$$

which is just the MKG equation in the γ-metric.

We gauge-transformed KG into MKG but broke the space-time symmetry in the additional process of locally specializing in $\theta = \frac{Mc^2}{\hbar}t$. The mass plays a critical role, causing

TABLE I. Quantum wave equations and corresponding forms of energy-momentum relation.

In γ or $1/\gamma$-metric	Quantum wave equation	Corresponding form of energy-momentum relation
Schrödinger	$\nabla^2\Psi = -i\frac{2M}{\hbar}\frac{\partial}{\partial t}\Psi$; Eq. (5)	$T = \frac{P^2}{2M}$; Eq. (4) or $\mathscr{T} \approx \frac{\mathscr{P}^2}{2M}$; Eq. (6)
KG (γ)	$\partial_\mu\partial^\mu\Phi + \left(\frac{Mc}{\hbar}\right)^2\Phi = 0$	$E^2 = P^2c^2 + M^2c^4$; Eq. (1)
KG ($1/\gamma$)	$\partial_\mu\partial^\mu\Phi + \left(\frac{Mc}{\hbar\gamma}\right)^2\Phi = 0$	$\mathscr{E}^2 = \mathscr{P}^2c^2 + \left(\frac{M}{\gamma}\right)^2c^4$; Eq. (2)
MKG (γ)	$\partial_\mu\partial^\mu\Phi = i\frac{2M}{\hbar}\frac{\partial}{\partial t}\Phi$; Eq. (9)	$T^2 + 2Mc^2T = P^2c^2$; Eq. (7)
MKG ($1/\gamma$)	$\partial_\mu\partial^\mu\Phi = i\frac{2M}{\hbar\gamma}\frac{\partial}{\partial t}\Phi$; Eq. (11)	$\mathscr{T}^2 + \frac{2Mc^2}{\gamma}\mathscr{T} = \mathscr{P}^2c^2$
Dirac(γ)	$i\hbar\gamma^\mu\partial_\mu\Psi - Mc\Psi = 0$; Eq. (31)	$\gamma^\mu P_\mu - Mc = 0$; Eq. (21)
Dirac($1/\gamma$)	$i\hbar\gamma^\mu\partial_\mu\Psi - \frac{Mc}{\gamma}\Psi = 0$	$\gamma^\mu\mathscr{P}_\mu - \frac{Mc}{\gamma} = 0$
MDirac 1 (γ)	$\pm i\hbar\gamma^\mu\partial_\mu\Psi + (\gamma^0 - 1)Mc\Psi = 0$	$\gamma^\mu P_\mu + (\gamma^0 - 1)Mc = 0$
MDirac 1 ($1/\gamma$)	$\pm i\hbar\gamma^\mu\partial_\mu\Psi + (\gamma^0 - 1)\frac{Mc}{\gamma}\Psi = 0$; Eq. (28)	$\gamma^\mu\mathscr{P}_\mu + (\gamma^0 - 1)\frac{M}{\gamma}c = 0$; Eq. (25) (concise form)
MDirac 2 ($1/\gamma$)	$\pm i\hbar\gamma^\mu\partial_\mu\Psi + (\gamma^0 + 1)\frac{Mc}{\gamma}\Psi = 0$; Eq. (30)	$\gamma^\mu\mathscr{P}_\mu + (\gamma^0 + 1)\frac{M}{\gamma}c = 0$

spontaneous symmetry-breaking. We can reverse the procedure to transform MKG to KG to recover the space-time symmetry.

The relativistic extension of Schrödinger equation may naturally entail symmetry-breaking because the Schrödinger equation itself is symmetry-broken. In this sense, MKG is a straightforward extension of the Schrödinger equation while KG is a special, covariant version of the former. In both cases, the global symmetry is still preserved as they all obey (eventually) the same energy momentum relation, Eq. (1). This is analogous to the spontaneous symmetry-breaking in the BEH mechanism,[16,17] a feature enabled by the special field, the Higgs field.

C. Derivation of MDirac by a gauge transformation of Dirac

Similarly, the MDirac may be derived by a local U(1) gauge transformation of the Dirac equation according to $\Psi \to e^{\pm i\theta}\Psi$; $\theta = \theta(x)$. The first derivative is

$$\partial_\mu(e^{\pm i\theta}\Psi) = \pm i\partial_\mu\theta e^{\pm i\theta}\Psi + e^{\pm i\theta}\partial_\mu\Psi.$$

By plugging the above into the Dirac equation, Eq. (31), we then get

$$\mp\hbar\gamma^\mu(\partial_\mu\theta)\Psi + i\hbar\gamma^\mu\partial_\mu\Psi - Mc\Psi = 0.$$

They are the local U(1) gauge transformed Dirac equations. Now suppose $\theta = \theta(t) = \frac{Mc}{\hbar}x_0 = \frac{Mc^2}{\hbar}t$, then $\gamma^\mu\partial_\mu\theta = \gamma^0\partial_0\theta = \frac{Mc}{\hbar}\gamma^0$.

We then immediately arrive at

$$i\hbar\gamma^\mu\partial_\mu\Psi - (\gamma^0 + 1)Mc\Psi = 0$$
$$i\hbar\gamma^\mu\partial_\mu\Psi + (\gamma^0 - 1)Mc\Psi = 0,$$

which is the second of MDirac Eq. (30) and the first of MDirac Eq. (28), respectively. The latter also reads the first of Eq. (29) if Ψ is decoupled into the spinors, Ψ_A and Ψ_B. The Dirac equation has transformed into MDirac equations. In the original derivation, γ^0 was introduced for mathematical convenience *ad hoc*. Here, it shows up as a result of the gauge transformation specialized in the time dimension, a remarkable result. Owing to the second set of factored energy-momentum equation, Eq. (22), Ψ_A and Ψ_B in Eq. (29) may be interchanged. Each set is asymmetric but the global symmetry is preserved. They eventually obey the energy-momentum relation, Eq. (1).

D. Probability density of the modified Klein-Gordon wave function

It is well known that the wave functions of the Schrödinger equation allow the probability interpretation, while those of the KG equation do not. This section is extracted from Ref. 15, Sec. VII C to show the probability density of the MKG equation. For two real fields, ϕ_1 and ϕ_2, we consider a complex field, $\Phi = \phi + i\phi_2$, with a Lagrangian

$$\mathscr{L} = \partial_\mu\Phi^*\partial^\mu\Phi + i\alpha I_0^\mu\Phi^*(\partial_\mu\Phi),$$

of which the Euler-Lagrange equations are

$$\text{EL1}: \partial_\mu\partial^\mu\Phi = i\alpha I_0^\mu\partial_\mu\Phi, \quad \text{and}$$
$$\text{EL2}: \partial_\mu\partial^\mu\Phi^* = -i\alpha I_0^\mu\partial_\mu\Phi^*.$$

With $\alpha \equiv \frac{2Mc}{\hbar\gamma}$, EL1 in the above is Eq. (15); EL2 represents its antiparticle.

Now $\Phi^*\times$ EL1-$\Phi\times$ EL2 gives

$$\partial_\mu(\Phi^*\partial^\mu\Phi - \Phi\partial^\mu\Phi^*) = i\alpha\partial_0(\Phi^*\Phi).$$

In the LHS bracket is the Noether current which is conserved, hence in the RHS $\Phi^*\Phi$ must be constant in time. Thus $\Phi^*\Phi = \phi_1^2 + \phi_2^2$, being a positive definite constant, may be interpreted as the probability density (with suitable normalization.)

E. Renormalization

This section is taken from Ref. 15, Sec. VII D. Renormalization is an essential feature of the quantum field theory. For instance, the amplitudes of a one-loop Feynman diagram in Quantum Electrodynamics involve logarithmic divergence roughly in the form

$$\int^{\infty} \frac{dP}{P} = \ lnP|^{\infty} = \infty.$$

But in the $1/\gamma$-metric formulation, γ is embedded in the mass term represented by M/γ and the above integral would then appear as

$$\int^{Mc} \frac{d\mathscr{P}}{\mathscr{P}} = \int^{Mc} \frac{d(Mv)}{Mv} = \int^{c} \frac{dv}{v} = ln(|v|)|^{c} + Const.$$

Thus, the $1/\gamma$-metric formulation potentially removes the γ-divergence *a priori* and renders renormalization unnecessary, at least in the above sense.

V. CONCLUSION

A relativistic extension of the Schrödinger equation is shown to provide corrections to both the KG and Dirac equations. The new equation is the result of applying the quantum prescriptions to the momentum and the kinetic energy rather than to the momentum and the total (internal and kinetic) energy in the relativistic energy-momentum equation. It is justified since after all it is the kinetic energy that generates the momentum. The equation is written in both γ-metric and $1/\gamma$-metric with the latter shown to allow us to avoid infinity as the velocity of the particle approaches c while providing us with crucial (speed) information for the particle.

The "Min equation" is shown to reduce to the Schrödinger equation in the nonrelativistic limit. It is also shown to decouple into a modified form of the Dirac equation.

When applied to a rest particle, the modified Dirac equation not only describes spin one-half particles but also presents a vacuum state solution eliminating the need for the Dirac Sea. This vacuum state solution does not exist in the Dirac equation. The new particle solutions have the characteristic frequency that is twice that of the Dirac equation. In the plane wave solutions, the bispinors of the modified Dirac equation are shown to interact with each other through mass on the one hand and in the purely electromagnetic manner on the other. The bispinors of the Dirac equation interact with each other only through mass.

ACKNOWLEDGMENTS

The author thanks the anonymous reviewer for the valuable comments on the Lorentz-(co)variance of the new, relativistic quantum wave equations. Section IV is added in response to those comments.

[1]C. Itzykson and J. Zuber, *Quantum Field Theory*, Dover edition (2005, copyright by McGraw-Hill, Inc., New York, 1980).
[2]A. Einstein, *On the Electrodynamics of Moving Bodies*, English translation of the original 1905 German-language paper [published as Zur Elektrodynamik bewegter Körper, in *Ann. Phys.* 17, 891 (1905)] in *The Principle of Relativity* (Methuen and Company, Ltd., London, 1923).
[3]A. Einstein, "On a heuristic point of view about the creation and conversion of light," in *The Old Quantum Theory* (Pergamon Press, New York, 1967), pp. 91–107.
[4]J. Aharoni, *The Special Theory of Relativity* (Oxford University Press, New York, 1965).
[5]M. Born, *Einstein's Theory of Relativity* (Dover Publications, Mineola, NY, 1965).
[6]L. de Broglie, doctoral thesis of 1924, *Recherches sur la theorie des quanta* [Ann. de Phys., 10e série, t. III (Janvier-Février 1925)]. A translation by A. F. Kracklauer (2004) http://aflb.ensmp.fr/LDB-oeuvres/De_Broglie_Kracklauer.pdf.
[7]A. Einstein, *Relativity: The Special and General Theory*, translated by Robert W. Lawson and originally published by H. Holt, 1920 (Dover Publications, Mineola, NY, 2001), Section XV, p. 52.
[8]E. Schrödinger, Phys. Rev. **28**, 1049 (1926).
[9]P. A. M. Dirac, Proc. R. Soc. A: Math., Phys. Eng. Sci. **117**, 610 (1928).
[10]P. A. M. Dirac, Proc. R. Soc. A: Math., Phys. Eng. Sci. **126**, 360 (1928).
[11]H. Kragh, Ann. Phys. **9**, 961 (2000).
[12]S. Weinberg, *Lectures on Quantum Mechanics* (Cambridge University Press, New York, 2013.)
[13]D. Griffiths, *Introduction to Elementary Particles*, 2nd revised edition, (Wiley-VCH Verlag GmbH & Co. KGaA, Weinheim, Germany, 2010).
[14]M. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory* (Westview Press, Boulder, CO, 1995).
[15]B. B. K. Min, Phys. Essays **31**, 265 (2018).
[16]F. Englert, BEH mechanism its scalar boson (Nobel lecture, 2013), available online at https://www.nobelprize.org/prizes/physics/2013/englert/lecture/.
[17]P. W. Higgs, Evading Goldstone theorem (Nobel lecture, 2013), available online at https://www.nobelprize.org/prizes/physics/2013/higgs/lecture/.